

Entire functions and compact operators with S_p -imaginary component

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Abstract

We study operators of the form $X + Y$ where Y has a finite p -th Schatten norm, $1 \leq p < 2$, and X is selfadjoint and of Hilbert-Schmidt class. Our study is based on new theorems on zero distribution of entire functions of finite order.

Introduction

Let S_p be the von Neumann-Schatten class of compact operators in a Hilbert space \mathcal{H} such that

$$\|A\|_p = \left\{ \sum_k s_k^p(A) \right\}^{1/p} < \infty,$$

where $\{s_k(A)\}$ is a sequence of singular values of the operator A , that is eigenvalues of a non-negative operator $\sqrt{A^*A}$, enumerated in non-increasing order: $s_1(A) \geq s_2(A) \geq \dots$. If $p \geq 1$, this defines a norm, and for these p 's the classes S_p are Banach spaces. In particular, S_1 is the trace class, and S_2 is the Hilbert-Schmidt class.

For any operator A , we denote by

$$G = \frac{1}{2}(A + A^*) \quad \text{and} \quad H = \frac{1}{2i}(A - A^*)$$

its Hermitian parts.

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If $A \in S_2$, then we define the Carleman determinant of A as a canonical product of genus one

$$C_A(z) = \prod_k E(z\mu_k(A)), \quad E(z) = (1-z)e^z,$$

where $\{\mu_k(A)\}$ is the set of eigenvalues of A counted their multiplicities.

If $A = G + iH$ is a compact quasinilpotent operator, then, by a theorem of L. Sakhnovich [18], $\|G\|_2 = \|H\|_2$, and $\|A\|_2 \leq 2\|H\|_2$. Soon after Sakhnovich obtained his result, M. Krein discovered in [11] a weak type inequality for compact quasinilpotent operators with imaginary component of trace class (see inequality (1.6) below), and the first-named author proved in [14] that, for $1 < p < \infty$, $\|A\|_p \leq K_p\|H\|_p$. This can be regarded as boundedness of the operator of triangular integration (which recovers a compact quasinilpotent operator by its imaginary part) in the spaces S_p , $1 < p < \infty$, and its “weak boundedness” in the space S_1 (cf. [2], [8], and [4, Chapter 4]). There are numerous interplays and interrelations between these results and inequalities of M. Riesz and Kolmogorov for conjugate harmonic functions (cf. [14], [8, Chapter 4], [1], and [6]).

Original proofs given in [11] and [14] were based on the entire function theory. Later, more transparent geometrical proofs have been found. Recently, working on [17], we realized that function theory methods work in a more general situation, when the absolute value of the Carleman determinant C_A is not too small on the real axis. (If the operator A is quasinilpotent, then $C_A \equiv 1$.) This required a further development of function theory methods which was done in [16]. Operator theory results of the present work may be regarded as non-commutative counterparts of those of [16]. Probably, these results are new even for finite matrices. Our approach also leads to new results on entire functions (Theorems 3, 5 and 7 below) which may be interesting on their own.

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1. Results and conjectures

First, introduce some notations. For an entire function $f(z)$, we denote

$$M(r, f) = \max_{|z|=r} |f(z)|,$$

then

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log M(r)}{r}$$

is the exponential type of f . By $n(r, f)$ we denote the counting function of zeros of $f(z)$: it is the number of zeros of $f(z)$ in the disk $\{|z| \leq r\}$ counting multiplicities.

An entire function $f(z)$ is said to be of Cartwright class, if its exponential type is finite ($0 \leq \sigma(f) < \infty$), and

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(t)|}{t^2 + 1} dt < \infty.$$

Further, we set

$$\alpha_p(f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |f(t)|}{|t|^{p+1}} dt, \quad \alpha(f) = \alpha_1(f),$$

by $\alpha_p(f^{-1})$ and $\alpha(f^{-1})$ we denote similar integrals with $\log^- |f|$ instead of $\log^+ |f|$. We also set

$$\gamma_p(f) = \sum_k \left| \operatorname{Im} \frac{1}{z_k} \right|^p, \quad \gamma(f) = \gamma_1(f),$$

where the sum is taken over the zero set of f (counting multiplicities). We allow to these quantities to be equal to $+\infty$.

By K we denote various positive numerical constants, by K_p we denote positive values which may depend on p only.

Theorem 1. *If $A \in S_2$, then*

$$\sigma(C_G) + \alpha(C_G) = \alpha(C_G^{-1}) \leq \|H\|_1 + \alpha(C_A^{-1}). \quad (1.1)$$

In particular, if the RHS of (1.1) is finite, then the Carleman determinant $C_G(z)$ is of Cartwright class. Using a theory of entire functions of Cartwright class (cf. [13], [10]), we obtain an additional information about distribution of eigenvalues $\{\mu_k(G)\}$ of G . For example,

- for each $s > 0$,

$$\operatorname{card} \{k : |\mu_k(G)| \geq s\} \leq \frac{K}{s} \left(\|H\|_1 + \alpha(C_A^{-1}) \right); \quad (1.2)$$

- if the RHS of (1.1) is finite, then there exist the limits

$$\lim_{k \rightarrow \pm\infty} k\mu_k(G) = \frac{2}{\pi} \sigma(C_G). \quad (1.3)$$

Here we assume that μ_k are numerated in the decreasing order, and the positive (negative) values of indices k correspond to the positive (negative) values of μ_k .

Applying estimate (1.2) and Lemma 5.1 from [8, Chapter III], we obtain a weak-type inequality for singular values of A :

$$\begin{aligned} \text{card}\{k : s_k(A) \geq s\} &\geq s \\ &\leq \text{card}\{k : |\mu_k(G)| \geq s/2\} + \text{card}\{k : |\mu_k(H)| \geq s/2\} \\ &\leq \frac{K}{s} (||H||_1 + \alpha(C_A^{-1})) . \end{aligned} \quad (1.4)$$

Now, we mention two special cases. First, assume that the operator A is quasinilpotent, that is $\text{spec}(A) = \{0\}$. Then $C_A(z) \equiv 1$, and $\alpha(C_A^{-1}) = 0$. By a theorem of L. Sakhnovich [18] (see also [8, Chapter I]), a quasinilpotent compact operator A with an imaginary component of trace class belongs to S_2 . We arrive at

Theorem 2. *If A is a compact quasinilpotent operator with imaginary component of trace class, then*

$$\sigma(C_G) + \alpha(C_G) = \alpha(C_G^{-1}) \leq ||H||_1 , \quad (1.5)$$

and

$$\text{card}\{k : s_k(A) \geq s\} \leq \frac{K}{s} ||H||_1 . \quad (1.6)$$

This result, in essence, coincides with a theorem of M. Krein [7, Chapter IV, Theorem 8.2] which states that $C_G(z)$ is of Cartwright class, $\sigma(C_G) \leq ||H||_1$, and relation (1.6) holds. Our estimate (1.5) is more precise, and this is important for some applications (cf. [17]). Our proof of Theorem 1 uses Krein's arguments. We also mention that for quasinilpotent operators, Krein computed the type $\sigma(C_G)$ using triangular truncation (cf. [8, Chapter VII]).

An interesting question appears here which we were unable to answer: *Let $\Pi(z)$ be an arbitrary canonical product of genus one with real zeros and of Cartwright class, whether there exists a compact quasinilpotent operator $A = G + iH$ with a trace class imaginary component H and such that $C_G = \Pi$?*

Now, we consider another special case. In the space $l^2 = l^2(\mathbb{N})$ we define the operator of multiplication

$$(A_w a)(k) = w(k)a(k) , \quad \lim_{k \rightarrow \infty} w(k) = 0 .$$

Then A_w is a compact operator, and its spectrum coincides with the set $\{w(k)\}$. The singular values of A_w are $\{|w(k)|\}$. If $w \in l^2$, then $A_w \in S_2$, and

$$C_{A_w}(z) = \prod_k E(zw_k)$$

is an arbitrary canonical product of genus one.

Theorem 3. *Let $\Pi(z)$ be a canonical product of genus one. Then*

$$\sigma(\Pi) + \alpha(\Pi) = 2 \max \{\gamma_+(\Pi), \gamma_-(\Pi)\} + \alpha(\Pi^{-1}), \quad (1.7)$$

where $\gamma_+(\Pi)$ (correspondingly, $\gamma_-(\Pi)$) is the sum $\sum |Im(z_k^{-1})|$ taken over zeros of Π lying in the upper (lower) half-plane.

Equation (1.7) states that if one of its sides is finite, than the other is finite as well, the both sides are equal, and in particular $\Pi(z)$ belongs to the Cartwright class. We also get a sharp estimate

$$\sigma(\Pi) + \alpha(\Pi) \leq 2\gamma(\Pi) + \alpha(\Pi^{-1}). \quad (1.8)$$

Estimate (1.8) yields uniform upper bounds for $\log M(r, \Pi)$ and for the counting function of zeros $n(r, \Pi)$ in terms of $\gamma(\Pi) + \alpha(\Pi^{-1})$ (cf [16, Theorem 2]).

The next result deals with compact operators with imaginary component from the class S_p , $p > 1$. It is motivated by a result of the first-named author [14] which states that *if an imaginary component of a compact quasinilpotent operator A belongs to S_p , $1 < p < \infty$, then $A \in S_p$, and*

$$\|A\|_p \leq K_p \|H\|_p. \quad (1.9)$$

Another motivation comes from our recent work [16].

Theorem 4. *Let $A \in S_2$. Then, for $1 < p < 2$,*

$$\|A\|_p \leq K_p \left(\|H\|_p + \alpha_p^{1/p} (C_A^{-1}) \right). \quad (1.10)$$

In the case $p = 2$, one almost immediately gets

$$\|A\|_2 \leq 2\|H\|_2,$$

provided that $A \in S_2$, and

$$\int_0^\infty \frac{\log^- |C_A(t)|}{|t|^3} dt < \infty.$$

Indeed, we have

$$\log |C_A(t)| = -\frac{t^2}{2} \operatorname{Re} \operatorname{tr} A^2 + O(t^3) = \frac{t^2}{2} (\operatorname{tr} H^2 - \operatorname{tr} G^2) + O(t^3), \quad t \rightarrow 0,$$

and since the integral over a neighbourhood of the origin converges, the coefficient by t^2 must be non-negative:

$$\operatorname{tr}(H^2) \geq \operatorname{tr}(G^2),$$

whence

$$\|A\|_2 \leq \|G\|_2 + \|H\|_2 \leq 2\|H\|_2.$$

As special cases of Theorem 4, we obtain estimate (1.9) for compact quasinilpotent operators (in this case, by a duality argument [8], [4], the case $p > 2$ follows from the case $1 < p < 2$); and the following result on entire functions:

Theorem 5. *Let $\Pi(z)$ be a canonical product of genus one. Then, for $1 < p < 2$,*

$$\int_0^\infty \frac{\log M(r, \Pi)}{r^{p+1}} dr \leq K_p (\gamma_p(\Pi) + \alpha_p(\Pi^{-1})). \quad (1.11)$$

In reality, we first prove Theorem 5 and then, with its help, we establish Theorem 4.

A precise value of the best possible constant K_p in the inequality

$$\|G\|_p \leq K_p \|H\|_p, \quad 1 < p < \infty,$$

valid for compact quasinilpotent operators is still unknown for the most of values of p . There is a long-standing conjecture that it coincides with the value of the best possible constant in the inequality of M. Riesz found by Pichorides:

$$K_p = \begin{cases} \tan(\pi/2p) & \text{if } 1 < p \leq 2 \\ \cot(\pi/2p) & \text{otherwise.} \end{cases}$$

This is confirmed only for $p = 2^n$ and $p = 2^n/(2^n - 1)$, $n \in \mathbf{N}$ [9]. A value of the best possible constant in a more general inequality (1.10) is also unknown.

We cannot extend Theorems 4 to other normed ideals of compact operators (cf. [7]). For example, a natural conjecture is that, for $1 < p < 2$,

$$\sup_{n \in \mathbf{N}} \left\{ s_n(A) n^{1/p} \right\} \leq K_p \sup_{n \in \mathbf{N}} \left\{ s_n(H) n^{1/p} \right\}$$

provided that $A \in S_2$ and $|C_A(x)| \geq 1$ for $x \in \mathbf{R}$.

In Theorem 4, we assumed that $\alpha_p(C_A^{-1}) < \infty$, that is the function $x \mapsto \log |C_A(x)|$ is “almost non-negative” on the real line. In this form, the theorem cannot be extended to $p > 2$, as an example suggested by Nazarov shows (cf. [16]). However, there is another possible way to extend inequality (1.9) to more general classes of compact operators than quasinilpotent ones. Following Levin [12], we define a regularized determinant $D_A(z)$ of an arbitrary compact operator A as an entire function whose zero set coincides with the sequence $\{\mu_k^{-1}(A)\}$, and such that $D_A(0) = 1$. Naturally, such a choice is not unique. Now, we require that the function $\log^- |D_A(z)|$ is relatively small in \mathbf{C} :

$$\int_0^\infty \frac{m(r, D_A^{-1})}{r^{p+1}} dr < \infty,$$

where

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

is the Nevanlinna proximity function.

Theorem 6. *Let A be a compact operator, and let D_A be a regularized determinant of A . Then, for $1 < p \leq 2$,*

$$\|A\|_p^p \leq K_p \left(\|H\|_p^p + \int_0^\infty \frac{m(r, D_A^{-1})}{r^{p+1}} dr \right).$$

Observe, that we do not require here that A is a Hilbert-Schmidt class operator. The proof of this result is similar to that of Theorem 4, but is more lengthy, and we will not give it in this paper. We are unable to extend Theorem 6 to the case $p > 2$, such an extension seems plausible. However, the corresponding result from the entire function theory holds for the whole range $1 < p < \infty$ (cf. [16, Theorem 5]):

Theorem 7. *Let $f(z)$ be an entire function such that $f(0) = 1$. Then for $1 < p < \infty$,*

$$\int_0^\infty \frac{\log M(r, f)}{r^{p+1}} dr \leq K_p \left(\gamma_p(f) + \int_0^\infty \frac{m(r, 1/f)}{r^{p+1}} dr \right).$$

The last remark is that it seems to be important to have “prototheorems” which would contain the results of this work and our previous work [16] as special cases. A natural context for such prototheorems is provided by von Neumann operator algebras with finite or semi-finite traces (cf. [5]). For estimates (1.9) and (1.6) such generalizations are known (cf. [1] and [6]). In [3], L. Brown introduced subharmonic counterparts of the regularized determinants for operators which belong to one of non-commutative $L^p(M, \tau)$ classes, where M is a von Neumann algebra and τ is a faithful, normal, semi-finite trace on M . Using his results and the arguments given below, one can prove a rather general version of Theorem 1. Probably, a similar version of Theorem 4 also holds.

2. Proofs of Theorems 1 and 3

Proof of Theorem 1: Let P_\pm be the orthogonal projectors onto subspaces generated by those of eigenfunctions of H which correspond to non-negative (negative) eigenvalues. Let $H_\pm = \pm H P_\pm$. The operators H_\pm are non-negative, and $H = H_+ - H_-$. Set $H_1 = H_+ + H_-$. Then eigenvalues of H_1 are absolute values of eigenvalues of H and $\|H\|_1 = \text{tr}(H_1)$.

Introduce an auxiliary operator $A_1 = G + iH_1$. Since $G - A \in S_1$ and $A - A_1 \in S_1$, we can define the perturbation determinants:

$$\Delta_{G/A_1}(z) = \det \left[(I - zG)(I - zA_1)^{-1} \right] = \frac{C_G(z)}{C_{A_1}(z)} e^{iz \text{tr}(H_1)}, \quad (2.1)$$

and

$$\Delta_{A_1/A}(z) = \det \left[(I - zA_1)(I - zA)^{-1} \right] = \frac{C_{A_1}(z)}{C_A(z)} e^{iz \text{tr}(H - H_1)} \quad (2.2)$$

(cf. [7, Chapter IV]). Combining (2.1) and (2.2), we obtain

$$C_G(z) = \Delta_{G/A_1}(z) \Delta_{A_1/A}(z) C_A(z) e^{-iz \text{tr}(H)}. \quad (2.3)$$

A chief fact here is a result which goes back to M. Livšic [7, Chapter IV, Theorem 5.2]: *if $A = G + iH$ is compact and dissipative (that is, $H \geq 0$),*

and $\text{tr}(H) < \infty$, and if $B = G + iF$ is compact and $-H \leq F \leq H$, then $|\Delta_{B/A}(z)| \leq 1$, for z in the upper half-plane.

Applying twice this theorem, we obtain

$$|\Delta_{G/A_1}(z)| \leq 1, \quad \text{Im}z \geq 0, \quad (2.4)$$

and

$$|\Delta_{A_1/A}(z)| = |\Delta_{A/A_1}(z)|^{-1} \geq 1, \quad \text{Im}z \geq 0. \quad (2.5)$$

Thus, by virtue of (2.3),

$$\log^- |C_G(t)| \leq \log \frac{1}{|\Delta_{G/A_1}(t)|} + \log^- |C_A(t)|,$$

and

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^- |C_G(t)|}{t^2} dt \leq -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log |\Delta_{G/A_1}(t)|}{t^2} dt + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^- |C_A(t)|}{t^2} dt. \quad (2.6)$$

Now, we estimate the first integral in the RHS of (2.6). Since $\log |\Delta_{G/A_1}(z)|$ is a non-positive harmonic function in the upper half-plane,

$$\log |\Delta_{G/A_1}(iy)| = cy + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |\Delta_{G/A_1}(t)|}{t^2 + y^2} dt$$

with $c \leq 0$. Whence

$$\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |\Delta_{G/A_1}(t)|}{t^2 + y^2} dt \geq \lim_{y \downarrow 0} \frac{\log |\Delta_{G/A_1}(iy)|}{y}.$$

The latter limit can be computed using relation (2.1): since C_{A_1} and C_G are canonical products of genus one,

$$\log |C_{A_1}(z)| = O(z^2), \quad \text{and} \quad \log |C_G(z)| = O(z^2),$$

for $z \rightarrow 0$, and

$$\lim_{y \downarrow 0} \frac{\log |\Delta_{G/A_1}(iy)|}{y} = -\text{tr}(H_1).$$

Returning back to (2.6), we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^- |C_G(t)|}{t^2} dt \leq \|H\|_1 + \alpha(C_A^{-1}). \quad (2.7)$$

Next, we prove that, for each $\epsilon > 0$,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log |C_G(t)|}{t^2} e^{-\epsilon t^2} dt \leq 0. \quad (2.8)$$

Indeed, denoting by $\{\xi_k\}$ the zero set of C_G , and integrating termwise the formula

$$\log |C_G(t)| = \sum_k \log |E(t/\xi_k)|,$$

we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log |C_G(t)|}{t^2} e^{-\epsilon t^2} dt &= \sum_k \int_{-\infty}^{\infty} \frac{\log |E(t/\xi_k)|}{t^2} e^{-\epsilon t^2} dt \\ &= \sum_k \int_0^{\infty} \frac{\log |1 - t^2/\xi_k^2|}{t^2} e^{-\epsilon t^2} dt \\ &= \sum_k \left[\int_0^{\sqrt{2}|\xi_k|} + \int_{\sqrt{2}|\xi_k|}^{\infty} \right] \frac{\log |1 - t^2/\xi_k^2|}{t^2} e^{-\epsilon t^2} dt \\ &\leq \sum_k e^{-2\epsilon\xi_k^2} \int_0^{\infty} \frac{\log |1 - t^2/\xi_k^2|}{t^2} dt \\ &= 0, \end{aligned}$$

since

$$\int_0^{\infty} \frac{\log |1 - t^2|}{t^2} dt = 0.$$

This proves (2.8).

Now, we obtain that, for each $\epsilon > 0$,

$$\begin{aligned} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^+ |C_G(t)|}{t^2} e^{-\epsilon t^2} dt &\stackrel{(2.8)}{\leq} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log^- |C_G(t)|}{t^2} e^{-\epsilon t^2} dt \\ &\stackrel{(2.7)}{\leq} \|H\|_1 + \alpha(C_A^{-1}), \end{aligned}$$

and therefore, by the monotone convergence theorem,

$$\alpha(C_G) \leq \|H\|_1 + \alpha(C_A^{-1}). \quad (2.9)$$

It remains to improve a little bit estimate (2.9): in its LHS we must replace $\alpha(C_G)$ by $\sigma(C_G) + \alpha(C_G)$.

First, observe that since $C_G(z)$ has a minimal type with respect to order two, we have, for z in the upper half-plane,

$$\log |C_G(z)| = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |C_G(t)|}{|t - z|^2} dt + c_0 + c_1 \operatorname{Im} z + c_2 \operatorname{Im} z^2. \quad (2.10)$$

The Poisson integral in the RHS of (2.10) is absolutely convergent due to estimates (2.7) and (2.9). Putting in (2.10) $z = iy$, $y \downarrow 0$, we obtain that $c_0 = 0$. Putting there $z = re^{i\pi/4}$, $r \rightarrow \infty$, we obtain that $c_2 = 0$. At last, putting again $z = iy$, dividing the both sides by y , and letting $y \rightarrow 0$, we obtain

$$0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log |C_G(t)|}{t^2} dt + c_1,$$

or

$$c_1 + \alpha(C_G) = \alpha(C_G^{-1}).$$

On the other hand, we see from (2.10) (with $c_0 = c_2 = 0$) that

$$c_1 = \lim_{y \rightarrow \infty} \frac{\log |C_G(iy)|}{y} = \sigma(C_G).$$

This completes the proof of Theorem 1. \square

Proof of Theorem 3: We keep the previous notations which are applied now to the operator of multiplication $A = A_w$ defined above (before formulation of Theorem 3). Since we already know that $C_G(z)$ is of Cartwright class, it has a bounded type; i.e. is a quotient of bounded analytic functions in the upper half-plane. Then, using relations (2.3)–(2.5), we see that $\Pi(z) = C_A(z)$ also has a bounded type in the upper half-plane. The same argument shows, that $\Pi(z)$ has a bounded type in the lower half-plane. Applying a theorem of M. Krein (cf. [13, Lecture 16]), we obtain that $\Pi(z)$ is of Cartwright class.

We need some classical facts about entire functions of Cartwright class which may be found in [13, Lectures 16 and 17]. Let $f(z)$ be an entire function of Cartwright type. The quantities

$$\sigma_{\pm}(f) = \limsup_{y \rightarrow +\infty} \frac{\log |f(\pm iy)|}{y}$$

are called the types of f in the upper and lower half-planes correspondingly. Then

(i)

$$\sigma(f) = \max\{\sigma_+(f), \sigma_-(f)\};$$

(ii)

$$\lim_{R \rightarrow \infty} \frac{1}{\pi R} \int_0^\pi \log |f(Re^{i\theta})| \sin \theta d\theta = \frac{\sigma_+(f)}{\pi} \int_0^\pi \sin^2 \theta d\theta = \frac{\sigma_+(f)}{2};$$

Now, we apply the Carleman integral formula for the upper half-plane (cf. [13, Lecture 24]) to the function $\Pi(z)$. Due to condition $\log |\Pi(z)| = O(|z^2|)$ as $z \rightarrow 0$, it has no error term, and we get the relation

$$\begin{aligned} & 2 \sum_{0 < \theta_k < \pi} \left(\frac{1}{r_k} - \frac{r_k}{R^2} \right)^+ \sin \theta_k \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{t^2} - \frac{1}{R^2} \right)^+ \log |\Pi(t)| dt + \frac{2}{\pi R} \int_0^\pi \log |\Pi(Re^{i\theta})| \sin \theta d\theta, \end{aligned} \quad (2.11)$$

where $\{z_k\} = \{r_k e^{i\theta_k}\}$ are zeros of Π . Using the dominate convergence theorem and property (ii), we easily make in (2.11) the limit transition for $R \rightarrow \infty$, and obtain the limit relation

$$2\gamma_+(\Pi) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\log |\Pi(t)|}{t^2} dt + \sigma_+(f),$$

or

$$\sigma_+(\Pi) + \alpha(\Pi) = 2\gamma_+(\Pi) + \alpha(\Pi^{-1}). \quad (2.12)$$

Applying the same argument in the lower half-plane, we obtain the second relation

$$\sigma_-(\Pi) + \alpha(\Pi) = 2\gamma_-(\Pi) + \alpha(\Pi^{-1}). \quad (2.13)$$

Relations (2.12), (2.13) and property (i) of entire functions of Cartwright class complete the proof of Theorem 3. \square

One can prove Theorem 3 directly, without appeal to Theorem 1, along similar lines. Then, instead of a deep result from operator theory due to Livšic, we need its “scalar version”: a simple inequality which concerns complex numbers: *if w_1, w_2 are complex numbers such that $\operatorname{Re} w_1 = \operatorname{Re} w_2$ and $|\operatorname{Im} w_1| \leq \operatorname{Im} w_2$, then, for all z in the upper half-plane,*

$$\left| \frac{1 - zw_1}{1 - zw_2} \right| < 1.$$

I. Ostrovskii recently proposed another (a little bit shorter) proof of Theorem 3 based on a different idea.

3. Proofs of Theorems 4 and 5

First, we prove Theorem 5 and then Theorem 4.

Proof of Theorem 5: Let $\{z_k\}$ be zeros of the canonical product $\Pi(z)$ of genus one. We prove that

$$\sum_k \frac{1}{|z_k|^p} \leq K_p (\gamma_p(\Pi) + \alpha_p(\Pi^{-1})). \quad (3.1)$$

Then a standard application of Borel's estimate of the canonical product (cf. [13, Lecture 4]) yields estimate (1.11):

$$\log M(r, \Pi) \leq Kr \left(\int_0^r \frac{n(t, \Pi)}{t^2} dt + r \int_r^\infty \frac{n(t, \Pi)}{t^3} dt \right),$$

whence

$$\begin{aligned} \int_0^\infty \frac{\log M(r, \Pi)}{r^{p+1}} dr &\leq K \int_0^\infty \frac{dr}{r^p} \left(\int_0^r \frac{n(t, \Pi)}{t^2} dt + r \int_r^\infty \frac{n(t, \Pi)}{t^3} dt \right) \\ &\leq K \int_0^\infty \frac{n(t, \Pi)}{t^2} dt \left(\frac{1}{t} \int_0^t \frac{dr}{r^{p-1}} + \int_t^\infty \frac{dr}{r^p} \right) \\ &\leq K_p \int_0^\infty \frac{n(t, \Pi)}{t^{p+1}} dt \\ &\stackrel{(3.1)}{\leq} K_p (\gamma_p(\Pi) + \alpha_p(\Pi^{-1})). \end{aligned}$$

Let $\theta_k = \arg z_k$, $-\pi \leq \theta_k \leq \pi$. Since, for $|\varphi| \leq \pi/2$ and $1 < p < 2$, $\cos p\varphi \leq K_p \cos^p \varphi$, we have

$$\sum_k \frac{\cos^+ p(|\theta_k| - \pi/2)}{|z_k|^p} \leq K_p \sum_k \frac{|\sin \theta_k|^p}{|z_k|^p} = K_p \gamma_p(\Pi). \quad (3.2)$$

Claim.

$$\sum_k \frac{\cos^- p(|\theta_k| - \pi/2)}{|z_k|^p} \leq \sum_k \frac{\cos^+ p(|\theta_k| - \pi/2)}{|z_k|^p} + K_p \alpha_p(\Pi^{-1}), \quad (3.3)$$

with

$$K_p = \frac{p \sin \frac{\pi p}{2}}{2\pi}.$$

In particular, the LHS of (3.3) is finite, provided that $\gamma_p(\Pi) + \alpha_p(\Pi^{-1}) < \infty$.

First, we show that the claim easily yields inequality (3.1), and then we shall prove the claim.

Fix η , $0 < \eta < \frac{\pi}{2} - \frac{\pi}{2p}$, and split the sum in the LHS of (3.1) into two parts. Into the first sum (which we denote \sum') we include those k 's that $\eta \leq |\theta_k| \leq \pi - \eta$, and in the second sum (denoted by \sum'') we include such k 's that either $|\theta_k| < \eta$, or $|\theta_k \pm \pi| \leq \eta$. Then

$$\sum'_k \frac{1}{|z_k|^p} \leq K_p \sum_k \left| \operatorname{Im} \frac{1}{z_k} \right|^p,$$

and, using (3.2) and (3.3),

$$\sum''_k \frac{1}{|z_k|^p} \leq K_p \sum_k \frac{\cos^- p(|\theta_k| - \pi/2)}{|z_k|^p} \leq K_p (\gamma_p(\Pi) + \alpha_p(\Pi^{-1})),$$

proving (3.1).

Proof of the Claim: For each $\epsilon > 0$, we have

$$\begin{aligned} -\alpha_p(\Pi^{-1}) &\leq \int_{-\infty}^{\infty} \frac{\log |\Pi(x)|}{|x|^{p+1}} e^{-\epsilon x^2} dx \\ &= \sum_k \int_{-\infty}^{\infty} \frac{\log |E(x/z_k)|}{|x|^{p+1}} e^{-\epsilon x^2} dx \\ &= \sum_k \int_0^{\infty} \frac{\log |1 - x^2/z_k^2|}{|x|^{p+1}} e^{-\epsilon x^2} dx \\ &= \sum_k \frac{1}{|z_k|^{p+1}} \int_0^{\infty} \frac{\log |1 - se^{-2i\theta_k}|}{s^{p/2+1}} e^{-\epsilon s|z_k|^2} ds. \end{aligned} \quad (3.4)$$

Now,

$$\begin{aligned} \int_0^{\infty} \frac{\log |1 - se^{-2i\theta}|}{s^{p/2+1}} e^{-\epsilon s\rho^2} ds &= \left(\int_0^{2\cos^+ 2\theta} + \int_{2\cos^+ 2\theta}^{\infty} \right) \frac{\log |1 - se^{-2i\theta}|}{s^{p/2+1}} e^{-\epsilon s\rho^2} ds \\ &\leq e^{-2\epsilon \rho^2 \cos^+ 2\theta} \int_0^{\infty} \frac{\log |1 - se^{-2i\theta}|}{s^{p/2+1}} ds, \end{aligned}$$

since the kernel $s \mapsto \log |1 - se^{-2i\theta}|$ is negative for $0 < s < 2\cos^+ 2\theta$ and positive for $2\cos^+ 2\theta < s < \infty$. Computing the latter integral:

$$\int_0^{\infty} \frac{\log |1 - se^{-2i\theta}|}{s^{p/2+1}} ds = \frac{\pi}{\frac{p}{2} \sin \frac{\pi p}{2}} \cos \frac{p}{2}(2|\theta| - \pi), \quad -\pi \leq \theta \leq \pi,$$

we get

$$\int_0^\infty \frac{\log|1 - se^{-2i\theta}|}{s^{p/2+1}} e^{-\epsilon s\rho^2} ds \leq \frac{\pi}{\frac{p}{2}\sin\frac{\pi p}{2}} e^{-2\epsilon\rho^2 \cos^+ 2\theta} \cos p(|\theta| - \pi/2). \quad (3.5)$$

Plugging (3.5) into (3.4) with $\theta = \theta_k$ and $\rho = |z_k|$, we obtain

$$\sum_k \frac{\cos^- p(|\theta_k| - \pi/2)}{|z_k|^p} e^{-2\epsilon|z_k|^2 \cos 2\theta_k} \leq \sum_k \frac{\cos^+ p(|\theta_k| - \pi/2)}{|z_k|^p} + \frac{p \sin \frac{\pi p}{2}}{2\pi} \alpha_p(\Pi^{-1}).$$

By the monotone convergence theorem applied for $\epsilon \rightarrow 0$, this proves the claim and completes the proof of the theorem. \square

Proof of Theorem 4 uses the strategy from [14] (see also [15, Chapter IV]) combined with Theorem 5. Define the resolvent of G

$$R_G(z) = (I - zG)^{-1}.$$

Since $H \in S_p$ and S_p is an ideal, $HR_G(z) \in S_p \subset S_2$, and therefore, for regular values of z , $[HR_G(z)]^2 \in S_1$. Hence we can define the determinant

$$D(z) = \det[I + z^2(HR_G(z))^2]. \quad (3.6)$$

Since

$$I + z^2[HR_G(z)]^2 = (I - zA^*)(I - zG)^{-1}(I - zA)(I - zG)^{-1}, \quad (3.7)$$

we obtain the factorization

$$D(z) = \frac{C_A(z)C_{A^*}(z)}{C_G^2(z)} \quad (3.8)$$

which plays an important role in our considerations. Formula (3.8) follows immediately, if $A \in S_1$; in the general case, we first approximate $A \in S_2$ by finite rank operators A_n , verify relation (3.8) for A_n , and then let $n \rightarrow \infty$.

Now, using relation (3.6), we derive a good upper bound for $|D(re^{i\theta})|$, $\theta \neq 0, \pi$. According to one of the equivalent definitions of the determinant (cf. [7, Chapter IV]), we have

$$D(z) = \prod_k \left(1 + z^2\mu_k^2(HR_G)\right),$$

where $\mu_k(HR_G)$ are eigenvalues of $HR_G(z)$. Then, using a corollary to Weyl's inequality (cf. [7, Chapter II, § 3]) we get

$$\begin{aligned} |D(re^{i\theta})| &\leq \prod_k \left(1 + r^2 |\mu_k(HR_G)|^2\right) \\ &\leq \prod_k \left(1 + r^2 s_k^2(HR_G)\right) \\ &\leq \prod_k \left(1 + r^2 \|R_G(re^{i\theta})\|^2 s_k^2(H)\right) \\ &\leq \prod_k \left(1 + \frac{r^2}{|\sin \theta|^2} s_k^2(H)\right), \end{aligned} \quad (3.9)$$

since $s_k(AB) \leq \|A\|s_k(B)$, and $\|R_G(re^{i\theta})\| \leq |\sin \theta|^{-1}$.

Let $\nu(t; H)$ be the counting function of the sequence $\{s_k^{-1}(H)\}$:

$$\nu(t; H) = \text{card} \left\{ k : \frac{1}{s_k(H)} \leq t \right\},$$

and let $\eta = r/|\sin \theta|$. Then, continuing estimate (3.9), we obtain

$$\log |D(re^{i\theta})| \leq \int_0^\infty \log \left(1 + \frac{\eta^2}{t^2}\right) d\nu(t; H) = 2\eta^2 \int_0^\infty \frac{\nu(t; H)}{t(t^2 + \eta^2)} dt. \quad (3.10)$$

We set

$$T(\eta) = \eta^2 \int_0^\infty \frac{\nu(t; H)}{t(t^2 + \eta^2)} dt.$$

This is an increasing function of η , such that

$$T(+0) = T'(+0) = 0, \quad (3.11)$$

and

$$\begin{aligned} \int_0^\infty \frac{T(\eta)}{\eta^{p+1}} d\eta &= \int_0^\infty \frac{\nu(t; H)}{t} dt \int_0^\infty \frac{d\eta}{\eta^{p-1}(t^2 + \eta^2)} \\ &= \left(\int_0^\infty \frac{ds}{s^{p-1}(s^2 + 1)} \right) \int_0^\infty \frac{\nu(t; H)}{t^{p+1}} dt \\ &= \frac{\pi}{2p \sin \frac{\pi p}{2}} \|H\|_p^p. \end{aligned} \quad (3.12)$$

Now, using factorization (3.8) and the upper bound (3.10), we obtain the lower bound for the function $C_G(z)$:

$$\log^- |C_G(re^{i\theta})| \leq \frac{\log^- |C_A(re^{i\theta})| + \log^- |C_{A^*}(re^{i\theta})|}{2} + T\left(\frac{r}{|\sin \theta|}\right). \quad (3.13)$$

Inequality (3.13) provides us with a lower bound for the function $\log |C_G(z)|$. Later, we perform it into an upper bound for that function, but first, we must prepare estimates for the RHS of (3.13). For the second term, we already have estimate (3.12). In order to estimate the first sum in the RHS of (3.13), we use Theorem 5.

According to Theorem 6.1 from [7, Chapter II],

$$\gamma_p(C_A) = \sum_k |\operatorname{Im} \mu_k(A)|^p \leq \sum_k |\mu_k(H)|^p = \|H\|_p^p.$$

Therefore, our Theorem 5 is applicable to the function $C_A(z)$, and

$$\int_0^\infty \frac{m(r, C_A)}{r^{p+1}} dr \leq \int_0^\infty \frac{\log M(r, C_A)}{r^{p+1}} dr \leq K_p \left(\|H\|_p^p + \alpha_p(C_A^{-1}) \right),$$

where

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

is the Nevanlinna proximity function. Since $C_A(0) = 1$, making use of Jensen's formula, we obtain $m(r, C_A^{-1}) \leq m(r, C_A)$, and therefore

$$\int_0^\infty \frac{m(r, C_A^{-1})}{r^{p+1}} dr \leq \int_0^\infty \frac{m(r, C_A)}{r^{p+1}} dr \leq K_p \left(\|H\|_p^p + \alpha_p(C_A^{-1}) \right). \quad (3.14)$$

Since zeros of C_{A^*} are conjugates to those of C_A , $m(r, C_{A^*}^{-1}) = m(r, C_A^{-1})$, so we have the same upper bound for $m(r, C_{A^*}^{-1})$.

Claim. *Estimate (3.13) together with bounds (3.10), (3.12) and (3.14) yield the upper bound*

$$\int_0^\infty \frac{\log |C_G(iy)|}{y^{p+1}} dy \leq K_p \left(\|H\|_p^p + \alpha_p(C_A^{-1}) \right). \quad (3.15)$$

First, assuming (3.15), we complete the proof of the theorem, and then we shall prove the claim.

Since C_G is a canonical product of genus one, we have for $\Im z \neq 0$

$$\log |C_G(z)| = \operatorname{Re} \int_{-\infty}^{\infty} \left[\log \left(1 - \frac{z}{t} \right) + \frac{z}{t} \right] dh_G(t),$$

where $h_G(t)$ equals the number of zeros of C_G on $(0, t]$, if $t > 0$, and equals the number of zeros of C_G on $[t, 0)$ with the minus sign, if $t < 0$. Integrating by parts in the RHS, we obtain

$$\log |C_G(z)| = \operatorname{Re} \left[z^2 \int_{-\infty}^{\infty} \frac{h_G(t)}{t^2(z-t)} dt \right],$$

whence

$$\log |C_G(iy)| = \operatorname{Re} \left[-y^2 \int_{-\infty}^{\infty} \frac{h_G(t)}{t^2(iy-t)} dt \right] = y^2 \int_{-\infty}^{\infty} \frac{h_G(t)}{t(t^2+y^2)} dt.$$

Now, we integrate the last relation by y and apply Fubini's theorem (the integrand is non-negative):

$$\begin{aligned} \int_0^{\infty} \frac{\log |C_G(iy)|}{y^{p+1}} dy &= \int_{-\infty}^{\infty} \frac{h_G(t)}{t} dt \int_0^{\infty} \frac{dy}{y^{p-1}(t^2+y^2)} \\ &= \frac{\pi}{2 \sin \frac{\pi p}{2}} \int_{-\infty}^{\infty} \frac{h_G(t)}{t|t|^p} dt. \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) we obtain

$$\int_{-\infty}^{\infty} \frac{h_G(t)}{t|t|^p} dt = \frac{2 \sin \frac{\pi p}{2}}{\pi} \int_0^{\infty} \frac{\log |C_G(iy)|}{y^{p+1}} dy \leq K_p \left(\|H\|_p^p + \alpha_p(C_A^{-1}) \right),$$

whence

$$\|G\|_p^p = \sum_k |\mu_k(G)|^p = \int_{-\infty}^{\infty} \frac{dh_G(t)}{|t|^p} = p \int_{-\infty}^{\infty} \frac{h_G(t)}{t|t|^p} dt \leq K_p \left(\|H\|_p^p + \alpha_p(C_A^{-1}) \right),$$

or $\|G\|_p \leq K_p \left(\|H\|_p + \alpha_p(C_A^{-1}) \right)$. Since $\|A\|_p \leq \|G\|_p + \|H\|_p$, we are done.

It remains to prove the claim. In the proof we use the following

Lemma. *Let $u(z)$ be a function harmonic in the upper half-plane, continuous up to the real axis, and such that*

$$|u(z)| = o(|z|), \quad z \rightarrow 0. \tag{3.17}$$

Then

$$u(re^{i\theta}) \leq \frac{K}{\sin \theta} \left\{ \frac{1}{\pi} \int_0^\pi u^-(2re^{i\varphi}) \sin \varphi d\varphi + \frac{r}{2\pi} \int_{-2r}^{2r} \frac{u^-(t)}{t^2} dt \right\}. \quad (3.18)$$

Since the proof of the lemma is a standard combination of R. Nevanlinna's representation of functions harmonic in the upper semi-disk and of Carleman's formula (cf. [15, Chapter I] and [13, Lecture 23]), we give it in the appendix.

Proof of the Claim: We fix δ , $1 < \delta < p$, set $\beta = (\pi - \pi/\delta)/2$, and define the function

$$u(z) = \log |C_G(z^{1/\delta} e^{i\beta})|.$$

This function is harmonic within the angle

$$(1 - \delta)\frac{\pi}{2} < \arg z < (1 + \delta)\frac{\pi}{2},$$

and, since $\log |C_G(z)| = O(|z|^2)$, $z \rightarrow 0$, it satisfies condition (3.17). We take an arbitrary ψ such that $|\psi| \leq (\delta - 1)\pi/4$ and apply estimate (3.18) to the function $u_\psi(z) = u(ze^{i\psi})$ choosing $\theta = \arg z = \pi/2 - \psi$. This way, we obtain an estimate for $\log |G(iy)| = u(iy^\delta)$:

$$\log |C_G(iy)| \leq K \left\{ \frac{1}{\pi} \int_0^\pi u^- \left(2y^\delta e^{i(\varphi+\psi)} \right) \sin \varphi d\varphi + y^\delta \int_{-2y^\delta}^{2y^\delta} \frac{u^-(te^{i\psi})}{t^2} dt \right\}.$$

Integrating by ψ and increasing, if needed, the intervals of integrations, we obtain

$$\log |C_G(iy)| \leq K_p \left\{ \frac{1}{\pi} \int_I u^- \left(2y^\delta e^{i\psi} \right) d\psi + y^\delta \int_0^{2y^\delta} \frac{dt}{t^2} \int_I u^-(te^{i\psi}) d\psi \right\},$$

where $I = [-(\delta - 1)\pi/4, \pi + (\delta - 1)\pi/4]$. Since

$$\int_I u^-(te^{i\psi}) d\psi = \int_J \log^- |C_G(t^{1/\delta} e^{i\eta})| d\eta, \quad (3.19)$$

where $J = [\beta_p, \pi - \beta_p]$ with $\beta_p > 0$, we obtain

$$\log |C_G(iy)| \leq K_p \left\{ \int_J \log^- |C_G(2^{1/\delta} ye^{i\eta})| d\eta + y^\delta \int_0^{2y^\delta} \frac{dt}{t^2} \int_J \log^- |C_G(t^{1/\delta} e^{i\eta})| d\eta \right\}.$$

Plugging here estimate (3.13), and using monotonicity of $T(\eta)$, we obtain

$$\begin{aligned} \log |C_G(iy)| &\leq K_p \left\{ m\left(2^{1/\delta}y, C_A^{-1}\right) + T\left(\frac{2^{1/\delta}y}{\sin \beta_p}\right) \right. \\ &\quad \left. + y^\delta \int_0^{2y^\delta} \frac{dt}{t^2} \left[m\left(t^{1/\delta}, C_A^{-1}\right) + T\left(\frac{t^{1/\delta}}{\sin \beta_p}\right) \right] \right\}. \end{aligned} \quad (3.20)$$

Integrating (3.20) with respect to y , we get

$$\begin{aligned} \int_0^\infty \frac{\log |C_G(iy)|}{y^{p+1}} dy &\leq K_p \left\{ \int_0^\infty \frac{m(y, C_A^{-1})}{y^{p+1}} dy + \int_0^\infty \frac{T(y)}{y^{p+1}} dy \right. \\ &\quad \left. + \int_0^\infty \frac{dy}{y^{1+p-\delta}} \int_0^{2y^\delta} \frac{dt}{t^2} \left[m\left(t^{1/\delta}, C_A^{-1}\right) + T\left(\frac{t^{1/\delta}}{\sin \beta_p}\right) \right] \right\}. \end{aligned} \quad (3.21)$$

Changing the order of integration, we obtain

$$\begin{aligned} &\int_0^\infty \frac{dy}{y^{1+p-\delta}} \int_0^{2y^\delta} \frac{m(t^{1/\delta}, C_A^{-1})}{t^2} dt \\ &= \int_0^\infty \frac{m(s, C_A^{-1})}{s^{\delta+1}} ds \int_{s2^{-1/\delta}}^\infty \frac{dy}{y^{1+p-\delta}} \leq K_p \int_0^\infty \frac{m(s, C_A^{-1})}{s^{p+1}} ds, \end{aligned} \quad (3.22)$$

and similarly

$$\int_0^\infty \frac{dy}{y^{1+p-\delta}} \int_0^{2y^\delta} \frac{dt}{t^2} T\left(\frac{t^{1/\delta}}{\sin \beta_p}\right) \leq K_p \int_0^\infty \frac{T(s)}{s^{p+1}} ds. \quad (3.23)$$

At last, plugging estimates (3.22) and (3.23) into (3.21), and using inequalities (3.11) and (3.13), we obtain (3.14), and complete the proof of the claim, and therefore, of Theorem 4. \square .

Observe that the same method based on the factorization (3.8) proves the following result which is probably known to the specialists:

Theorem 8. *Let A be a compact operator, and let $\mu(A) = \{\mu_k(A)\}$ be its sequence of eigenvalues counted with multiplicities. Then, for $1 < p \leq 2$,*

$$\|A\|_p \leq K_p (\|H\|_p + \|\mu(A)\|_{l^p}).$$

Appendix. Proof of the Lemma

According to the Nevanlinna-Green formula (cf. [13, Section 24.3]), for any harmonic function in the semi-disk $\{\text{Im}z \geq 0, |z| < R\}$ which is continuous up to the boundary,

$$\begin{aligned} u(z) &= \frac{R^2 - |z|^2}{2\pi} \int_0^\pi \left(\frac{1}{|Re^{i\varphi} - z|^2} - \frac{1}{|Re^{i\varphi} - \bar{z}|^2} \right) u(re^{i\varphi}) d\varphi \\ &\quad + \frac{\text{Im}z}{\pi} \int_{-R}^R \left(\frac{1}{|t - z|^2} - \frac{R^2}{|R^2 - tz|^2} \right) u(t) dt \\ &\leq \frac{R^2 - |z|^2}{2\pi} \int_0^\pi \left(\frac{1}{|Re^{i\varphi} - z|^2} - \frac{1}{|Re^{i\varphi} - \bar{z}|^2} \right) u^+(re^{i\varphi}) d\varphi \\ &\quad + \frac{\text{Im}z}{\pi} \int_{-R}^R \left(\frac{1}{|t - z|^2} - \frac{R^2}{|R^2 - tz|^2} \right) u^+(t) dt \end{aligned} \quad (A1)$$

since the both kernels are non-negative. Now, we estimate the kernels for $z = re^{i\theta}$ and $R = 2r$. For the first kernel $K_1(Re^{i\varphi}, z)$ we have

$$K_1(Re^{i\varphi}, z) \leq \frac{R^2 - r^2}{2\pi} \frac{4Rr \sin \theta \sin \varphi}{(R - r)^4} \leq \frac{12 \sin \varphi}{\pi}; \quad (A2)$$

and for the second kernel $K_2(t, z)$ we obtain

$$\begin{aligned} K_2(t, z) &\leq \frac{r \sin \theta}{\pi} \frac{R^2(R^2 - t^2)}{|t - z|^2 |R^2 - tz|^2} \\ &\leq \frac{r \sin \theta}{\pi} \frac{4(R^2 - t^2)}{t^2 R^2 \sin^2 \theta} \left(1 - \frac{|t|}{2R}\right)^{-2} \leq \frac{48r}{\pi \sin \theta} \left(\frac{1}{t^2} - \frac{1}{R^2}\right). \end{aligned} \quad (A3)$$

Plugging estimates (A2) and (A3) into (A1), we obtain

$$u(z) \leq \frac{12}{\pi} \int_0^\pi u^+(Re^{i\varphi}) \sin \varphi d\varphi + \frac{48r}{\pi \sin \theta} \int_{-R}^R \left(\frac{1}{t^2} - \frac{1}{R^2}\right) u^+(t) dt. \quad (A4)$$

Now, we use Carleman's formula (cf. [13, Lecture 24]). Due to condition (3.17), it has no error term, and therefore

$$\begin{aligned} &\frac{1}{\pi R} \int_0^\pi u^+(Re^{i\varphi}) \sin \varphi d\varphi + \frac{1}{2\pi} \int_{-R}^R \left(\frac{1}{t^2} - \frac{1}{R^2}\right) u^+(t) dt \\ &= \frac{1}{\pi R} \int_0^\pi u^-(Re^{i\varphi}) \sin \varphi d\varphi + \frac{1}{2\pi} \int_{-R}^R \left(\frac{1}{t^2} - \frac{1}{R^2}\right) u^-(t) dt. \end{aligned} \quad (A5)$$

Combining (A4) and (A5), we obtain (3.18) and complete the proof. \square

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